# RELATIONSHIPS WITHIN TRIANGLES 

Geometry 5
$\square$ This Slideshow was developed to accompany the textbook
$\square$ Larson Geometry
$\square$ By Larson, R., Boswell, L., Kanold, T. D., \& Stiff, L.

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$\square$ Some examples and diagrams are taken from the textbook.

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### 5.1 Midsegment Theorem and Coordinate Proof

$\square$ Draw a triangle in your notes
$\square$ Find the midpoints of two of the sides using a ruler
$\square$ Connect the midpoints of the two sides with a segment
$\square$ Measure the segment and the third side
$\square$ What do you notice?
$\square$ What else do you notice about those two segments?


Length should be $1 / 2$
They should be parallel

### 5.1 Midsegment Theorem and Coordinate Proof

$\square$ Midsegment of a Triangle
$\square$ Segment that connects the midpoints of two sides of a triangle

Midsegment Theorem
The midsegment of a triangle is parallel to the third side and is half as long as that side.

# 5.1 Midsegment Theorem and Coordinate Proof 

$\square$ Name the midsegments.
$\square$ Draw the third midsegment.

$\square$ Let UW be 81 inches. Find VS.

UV, WV

UW
$U W=1 / 2$ ST
$V T=1 / 2 S T$
$\mathrm{UW}=\mathrm{VT}=81$

### 5.1 Midsegment Theorem and Coordinate Proof

$\square$ Coordinate Proof

1. Use the origin as a vertex or center.
2. Place at least one side of the polygon on an axis.
3. Usually keep the figure within the first quadrant.
4. Use coordinates that make computations as simple as possible.
$\square$ You will prove things by calculating things like slope, distance, and midpoints


### 5.1 Midsegment Theorem and Coordinate Proof

$\square$ Place a square in a coordinate plane so that it is convenient for finding side lengths. Assign coordinates.


Place a right triangle in a coordinate $(0, b)$ plane so that it is convenient for finding side lengths. Assign coordinates.


### 5.1 Midsegment Theorem and <br> Coordinate Proof

$\square$ Write a coordinate proof of the midsegment theorem.

1. Draw the figure
2. Draw a midsegment.
3. Calculate the proof

a. Lines are parallel
b. $\quad$ Midsegment $=1 / 2$ third side

Since we're finding midpoints, it is convenient to use $2 a, 2 b$, and $2 c$ so that the midpoints are whole numbers.
Midpoint calculations:

$$
\begin{aligned}
& \text { midpoint }=\left(\left(x_{1}+x_{2}\right) / 2,\left(y_{1}+y_{2}\right) / 2\right) \\
& ((0+2 a) / 2,(0+0) / 2)=(a, 0) \\
& ((2 a+2 b) / 2,(0+2 c) / 2)=(a+b, c)
\end{aligned}
$$

Parallel calculations:

$$
\begin{aligned}
& \text { slope }=\left(y_{2}-y_{1}\right) /\left(x_{2}-x_{1}\right) \\
& m=(2 c-0) /(2 b-0)=c / b \\
& m=(c-0) /(a+b-a)=c / b
\end{aligned}
$$

slopes are the same so lines are parallel

Length calculation:

$$
\begin{aligned}
& \text { dist }=V\left(\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}\right) \\
& V\left((2 b-0)^{2}+(2 c-0)^{2}\right)=V\left(4 b^{2}+4 c^{2}\right)=V\left(4\left(b^{2}+c^{2}\right)\right)=2 V\left(b^{2}+c^{2}\right) \\
& V\left((a+b-a)^{2}+(c-0)^{2}\right)=V\left(b^{2}+c^{2}\right)
\end{aligned}
$$

### 5.1 Midsegment Theorem and Coordinate Proof

$\square 298$ \#2-1 8 even, 24-32 even, 36, 40, 42, 44, 4852 even $=21$ total
5.1 Answers and Quiz

- 5.1 Answers
- 5.1 Quiz


### 5.2 Use Perpendicular Bisectors

$\square$ Perpendicular Bisector
Segment that is perpendicular to and bisectsensegment

Perpendicular Bisector Theorem


If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment

Converse of the Perpendicular Bisector Theorem
If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment

### 5.2 Use Perpendicular Bisectors

In the diagram, $\overleftrightarrow{J K}$ is the perpendicular bisector of $\overline{N L}$.

Find NK.

- Explain why $M$ is on $\overleftrightarrow{J K}$.


Since JK is $\perp$ bisector, then NK $=\mathrm{LK}(\perp$ bisector theorem).
$6 x-5=4 x+1 \rightarrow 2 x-5=1 \rightarrow 2 x=6 \rightarrow x=3$
Find NK: $6 x-5 \rightarrow 6(3)-5=13$

Since $M N=M L, M$ is equidistant from each end of NL. Thus by then Converse of the Perpendicular Bisector Theorem, M is on the perpendicular bisector.

### 5.2 Use Perpendicular Bisectors

$\square$ Find the perpendicular bisectors of a triangle
$\square$ Cut out a triangle
$\square$ Fold each vertex to each other vertex
$\square$ The three folds are the perpendicular bisectors
$\square$ What do you notice?

- Perpendicular bisectors meet at one point
$\square$ Measure the distance from the meeting point to each vertex
$\square$ What do you notice?
- The distances are equal


### 5.2 Use Perpendicular Bisectors

$\square$ Concurrent
$\square$ Several lines that intersect at same point (point of concurrency)
Concurrency of Perpendicular Bisectors of a Triangle
The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of a triangle


### 5.2 Use Perpendicular Bisectors

$\square$ Hot pretzels are sold from store at $A, B$, and $E$. Where could the pretzel distributor be located if it is equidistant from those three points?


### 5.2 Use Perpendicular Bisectors

$\square$ Circumcenter
$\square$ The point of concurrency of the perpendicular bisectors of a triangle.
$\square$ If a circle was
circumscribed around a triangle, the circumcenter would also be the center of the circle.

5.2 Use Perpendicular Bisectors

■ 306 \#2-1 6 even, $20,22,26,28,30,34-40$ even
= 17 total
Extra Credit 309 \#2, $4=+2$
5.2 Answers and Quiz
$\square$ 5.2 Answers

- 5.2 Quiz


### 5.3 Use Angle Bisectors of Triangles



Angle Bisector
$\square$ Ray that bisects an angle


Angle Bisector Theorem
If a point is on the angle bisector, then it is equidistant from the sides of the angle

Converse of the Angle Bisector Theorem
If a point is equidistant from the sides of an angle, then it is on the angle bisector

### 5.3 Use Angle Bisectors of Triangles

- Find the value of $x$.

- Do you have enough information to conclude that $\overrightarrow{Q S}$ bisects $\angle P Q R$ ?

$3 x+5=4 x-6 \rightarrow 5=x-6 \rightarrow x=11$
$5 x=6 x-5 \rightarrow-x=-5 \rightarrow x=5$

No, you need to know that $\mathrm{SP} \perp \mathrm{QP}$ and $\mathrm{SR} \perp \mathrm{QR}$

### 5.3 Use Angle Bisectors of Triangles

## Concurrency of Angle Bisectors of a Triangle

The angle bisectors of a triangle intersect at a point that is equidistant from the sides of a triangle
$\square$ Incenter

- Point of concurrency of the angle bisectors of a triangle
- If a circle was inscribed in a triangle, the incenter would also be the center
 of the circle.


### 5.3 Use Angle Bisectors of Triangles

N is the incenter. Find EN .313 \#2-24 even, 30, 34, 40-46 even $=18$ total

Find NF by using the Pythagorean theorem.
$16^{2}+\mathrm{NF}^{2}=20^{2} \rightarrow 256+\mathrm{NF}^{2}=400 \rightarrow \mathrm{NF}^{2}=144 \rightarrow \mathrm{NF}=12$
Since $N$ is the incenter, $N F=E N=12$
5.3 Answers and Quiz

- 5.3 Answers
- 5.3 Quiz


### 5.4 Use Medians and Altitudes

Median
$\square$ Segment that connects a vertex to a midpoint of side of a triangle.
$\square$ Point of concurrency is called the centroid.

$\square$ The centroid is the balance point.
Concurrency of Medians of a Triangle
The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoints of the opposite side.

### 5.4 Use Medians and Altitudes

$\square$ Each path goes from the midpoint of one edge to the opposite corner. The paths meet at P.
$\square$ If $S C=2100 \mathrm{ft}$, find $P S$ and $P C$.
$\square$ If $B T=1000 \mathrm{ft}$, find $T C$ and $B C$.

$\square$ If PT $=800 \mathrm{ft}$, find PA and TA.
$P C=2 / 3 \mathrm{SC} \rightarrow \mathrm{PC}=2 / 3(2100)=1400 \mathrm{ft} \rightarrow \mathrm{PS}=700 \mathrm{ft}$
$T$ is midpoint of $B C . T C=1000 \mathrm{ft}, \mathrm{BC}=2000 \mathrm{ft}$

$$
\text { PT }=1 / 3 \mathrm{TA} \rightarrow 800=1 / 3 \mathrm{TA} \rightarrow 2400 \mathrm{ft}=\mathrm{TA}, \mathrm{PA}=1600 \mathrm{ft}
$$

### 5.4 Use Medians and Altitudes

## Altitudes

Segment from a vertex and perpendicular to the opposite side of a triangle.Point of concurrency is called the orthocenter.
Concurrency of Altitudes of a Triangle
The lines containing the altitudes of a triangle are concurrent.

There is nothing terribly interesting about the orthocenter. In an acute triangle, the orthocenter is inside the triangle. In a right triangle, the orthocenter is on the triangle at the right angle. In an obtuse triangle, the orthocenter is outside of the triangle.

### 5.4 Use Medians and Altitudes

$\square$ In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle are all the same segment.


### 5.4 Use Medians and Altitudes

$\square$ Find the orthocenter.


Draw the other two altitudes (from A and C). They will be outside the triangle.

OQ is also the perpendicular bisector, angle bisector, and median. P (-h, 0)

### 5.4 Use Medians and Altitudes

$\square \triangle \mathrm{PQR}$ is isosceles and segment $\overline{O Q}$ is an altitude. What else do you know about $\overline{O Q}$ ? What are the coordinates of $P$ ?


Draw the other two altitudes (from A and C). They will be outside the triangle.

OQ is also the perpendicular bisector, angle bisector, and median. P (-h, 0)

### 5.4 Use Medians and Altitudes

- Given: $\triangle \mathrm{ABC}$ is isosceles, $\overline{B D}$ is a median
- Prove: $\overline{B D}$ is an angle bisector


Statements Reasons
$\triangle A B C$ is isosceles, $B D$ is a median
$B A \cong B C$
$A D \cong D C$
$B D \cong B D$
$\triangle A B D \cong \triangle C B D$
$\angle A B D \cong \angle C B D$
$B D$ is an angle bisector
(given)
(def. Isosceles)
(def. Median)
(reflexive)
(def $\cong \Delta$ ) (CPCTC)
(def angle bisector)
5.4 Use Medians and Altitudes
$\square 322$ \#2-26 even, 34, 40, 42, 46-54 even $=20$ total
$\square$ Extra Credit 325 \#2, $4=+2$
5.4 Answers and Quiz

- 5.4 Answers
- 5.4 Quiz


### 5.5 Use Inequalities in a Triangle

$\square$ Draw a scalene triangle
$\square$ Measure the sides
$\square$ Measure the angles
$\square$ What do you notice?

$\square$ Smallest side opposite $\qquad$
$\square$ Largest angle opposite

Smallest angle
Largest side

### 5.5 Use Inequalities in a Triangle

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.
$\square$ List the sides in order from shortest to longest.


ST, RS, RT

### 5.5 Use Inequalities in a Triangle

$\square \quad$ Draw a triangle with sides $5 \mathrm{~cm}, 2 \mathrm{~cm}$, and 2 cm .
$\square$ Draw a triangle with sides $5 \mathrm{~cm}, 2 \mathrm{~cm}$, and 3 cm .
$\square$ Draw a triangle with sides $5 \mathrm{~cm}, 3 \mathrm{~cm}$, and 3 cm .
Triangle Inequality Theorem
The sum of two sides of a triangle is greater than the length of the third side.

$$
A B+B C>A C ; A B+A C>B C ; B C+A C>A B
$$

Can't be done, short side don't touch
Can't be done, forms a line
Can be done, isosceles triangle

### 5.5 Use Inequalities in a Triangle

$\square$ A triangle has one side of 11 inches and another of 15 inches. Describe the possible lengths of the third side.

$11+x>15 \rightarrow x>4$<br>$15+x>11 \rightarrow x>-4$ (not useful)<br>$11+15>x \rightarrow 26>x$<br>Combine $1^{\text {st }}$ and $3^{\text {rd }}$ : $4<x<26$

Short cut: subtract to get smallest, add to get largest

### 5.5 Use Inequalities in a Triangle

Find all possible values of $x$.

$\square 331$ \#2, 6-34 even, 40-44 even, 49, 52, $54=22$ total

$$
\begin{aligned}
& x+11+2 x+10>5 x-9 \rightarrow 3 x+21>5 x-9 \rightarrow 30>2 x \rightarrow 15>x \\
& 2 x+10+5 x-9>x+11 \rightarrow 7 x+1>x+11 \rightarrow 6 x>10 \rightarrow x>10 / 6 \rightarrow x>12 / 3 \\
& x+11+5 x-9>2 x+10 \rightarrow 6 x+2>2 x+10 \rightarrow 4 x>8 \rightarrow x>2
\end{aligned}
$$

Choose narrowest interval: $2<x<15$
5.5 Answers and Quiz

- 5.5 Answers
- 5.5 Quiz


### 5.6 Inequalities in Two Triangles and Indirect Proof

$\square$ See Mr. Wright's demonstration with the metersticks.
$\square$ How does the third side compare when there is a small angle to a big angle?

Use two meter sticks to demonstrate the SAS Inequality and SSS Inequality
Have two meter sticks form two sides of the $\Delta$ and have the kids imagine the third side.

### 5.6 Inequalities in Two Triangles and Indirect Proof

Hinge Theorem
If 2 sides of one $\Delta$ are congruent to 2 sides of another $\Delta$, and the included angle of the $1^{\text {st }} \Delta$ is larger than the included angle of the $2^{\text {nd }} \Delta$, then the $3^{\text {rd }}$ side of the $1^{\text {st }} \Delta$ is longer than the $3^{\text {rd }}$ side of the $2^{\text {nd }} \Delta$.


### 5.6 Inequalities in Two Triangles and Indirect Proof

Converse of the Hinge Theorem
If 2 sides of one $\Delta$ are congruent to 2 sides of another $\Delta$, and the $3^{\text {rd }}$ side of the first is longer than the $3^{\text {rd }}$ side of the $2^{\text {nd }} \Delta$, then the included angle of the $1^{\text {st }} \Delta$ is larger than the included angle of the $2^{\text {nd }} \Delta$.

5.6 Inequalities in Two Triangles and Indirect Proof
$\square$ If $P R=P S$ and $m \angle Q P R>m \angle Q P S$, which is longer, $\overline{S Q}$ or $\overline{R Q}$ ?

$\square$ If $P R=P S$ and $R Q<S Q$, which is larger, $m \angle R P Q$ or $\mathrm{m} \angle \mathrm{SPQ}$ ?

### 5.6 Inequalities in Two Triangles and Indirect Proof

$\square$ Indirect Reasoning

- You are taking a multiple choice test.
- You don't know the correct answer.
- You eliminate the answers you know are incorrect.
$\square$ The answer that is left is the correct answer.
$\square$ You can use the same type of logic to prove geometric things.


### 5.6 Inequalities in Two Triangles and Indirect Proof

$\square$ Indirect Proof
$\square$ Proving things by making an assumption and showing that the assumption leads to a contradiction.
$\square$ Essentially it is proof by eliminating all the other possibilities.

### 5.6 Inequalities in Two Triangles and Indirect Proof

$\square$ Steps for writing indirect proofs

- Identify what you are trying to prove. Temporarily, assume the conclusion is false and that the opposite is true.
- Show that this leads to a contradiction of the hypothesis or some other fact.
$\square$ Point out that the assumption must be false, so the conclusion must be true.


### 5.6 Inequalities in Two Triangles and Indirect Proof

- Example; Prove that through a point not on a line, there is only one line perpendicular to the line.

- We want to show that only one line is $\perp$ to $\overleftrightarrow{M N}$. So, assume that lines $P M$ and $\mathrm{PN} \perp$ to $\overleftrightarrow{M N}$.
- $\angle \mathrm{PMN}$ and $\angle \mathrm{PNM}$ are $\mathrm{rt} \angle \mathrm{s}$ by definition of $\perp$ lines
- $\mathrm{m} \angle \mathrm{PMN}=90^{\circ}, \mathrm{m} \angle \mathrm{PNM}=90^{\circ}$ by definition of right angles
- The angles of a triangle should add to be 180 degrees, but this triangle adds to more. This is a contradiction, so our assumption is false. And there is only one line perpendicular to the line through point $P$.


### 5.6 Inequalities in Two Triangles and Indirect Proof

$\square$ Suppose you wanted to prove the statement "If $x+$ $y \neq 14$ and $y=5$, then $x \neq 9$." What temporary assumption could you make to prove the conclusion indirectly?
$\square$ How does that assumption lead to a contradiction?

Assume $x=9$

If $x=9$, then $x+y \neq 14.9+5 \neq 14 \rightarrow 14 \neq 14$. This is the contradiction
5.6 Inequalities in Two Triangles and Indirect Proof
$\square 338$ \#4-18 even, 22-34 even $=15$ total
$\square$ Extra Credit 341 \#2, $4=+2$
5.6 Answers and Quiz

- 5.6 Answers
- 5.6 Quiz


## 5.Review

## CHAPIBRTIEST

## Two midsegments of $\triangle A B C$ are $\overline{D E}$ and $\overline{D F}$. <br> 1. Find DR <br> 2. Find $D F$ <br> 3. What can you conclude about $\overline{E N}$



## Find the value of $x$. Explain your reasoning.


7. In Exercise 4, is point $T$ on the perpendicular bisector of $\overline{\text { SU }}$ Explain
a. In the diagram at the right, the angle bisectors of $\triangle X Y Z$ meet at point $D$. Find $D R$.


In the diagram at the right, $P$ is the centroid of $\triangle R S T$.
9. If $L S=36$, find $P L$, and $P S$
10. If $T P=20$, find $T /$ and $P$.
11. $1 f / R=25$, find $/ S$ and $R S$.

12. Is it possible to coostruct a triangle with side leagths 9, 12, and 227 If not, explain why not.
13. In $\triangle A B C, A B=36, B C=18$, and $A C=22$. Sketch and label the triangle List the angles in order from smallest to largest

In the diagram for Exercises 14 and $15, ~ I L=M K$
14. If $m \angle J K M>m \angle L / K$ which is longes, $\overline{L K}$ or $\overline{M /}$ Explain
15. If MO < LK, which is larget, $\angle L J K$ or $\angle / K O M$ Explain.
16. Write a temporary assumption you could make to prove the


